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LETTER TO THE EDITOR

Number of two-choice and spiral self-avoiding loops

S S Manna

Saha Institute of Nuclear Physics, 92 Acharya Prafulla Chandra Road, Calcutta—700 009, India

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Abstract. The number of independent self-avoiding walk (SAW) loop configurations (C_N) for fixed perimeter length N on the square lattice is fitted here to the scaling form $C_N \sim \mu^N N^h$, for SAWs with various constraints. For two-choice SAWs, extrapolation of enumeration results gives $h = -1.5 \pm 0.1$ (which falls in the same universality class as ordinary SAWs) and exact calculations for spiral SAWs give $h = 3$.

Self-avoiding walks (SAWs) are random walks which do not visit one site more than once. Self-avoiding loops are closed loops with this self-avoiding restriction. It has been shown (de Gennes 1979) that both self-avoiding walks and loops are necessary to explain magnetism in the $n \to 0$ vector model. de Gennes showed that the number of independent SAW loops (C_N) with perimeter length N in the asymptotic region ($N \to \infty$ limit) can be fitted to a scaling form $C_N \sim \mu^N N^h$ where $h = -d\nu$, ν being the average end-to-end distance exponent and μ the average connectivity constant for the SAWs on a d-dimensional lattice.

Recently, interest has been shown in studying the effect of different constraints (microscopic or macroscopic) on SAW statistics, specifically in two dimensions. For example, Grassberger (1982) showed that two-choice SAWs, in which no two successive steps are allowed in the same lattice direction (microscopic constraint), belong to the same universality class as the ordinary SAW. In spiral SAWs (Privman 1983), the constraint is such that every step forbids its next step to be in the clockwise (or anticlockwise) direction, so that the walk spirals about a direction perpendicular to the plane of the walk. Such a macroscopic constraint also leads to a different universality class (Blöte and Hilhorst 1984, Guttmann and Wormald 1984) for statistics of SAWs.

Here, we have studied the asymptotic behaviour of self-avoiding loops with such microscopic constraints on the square lattice. The SAW loops with the two-choice constraint have been studied numerically and those with the spiral constraint have been done exactly.

To illustrate our result, first consider two-choice self-avoiding loops on the square lattice. It is seen that these types of loops can be found for perimeter lengths which are integral multiples of 4, except for length 8. Exact enumeration results for the number of independent loops with fixed perimeter length $N(C_N)$ up to 52† are given

† I am extremely grateful to Professor P Grassberger for supplying the results for the last four terms of this series.

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in table 1. To find the connectivity constant μ we followed Grassberger (1984) and defined

$$\mu_N = \left(\frac{(N+4)^x C_{N+4}}{N^x C_N}\right)^{1/4}.$$

Table 1. Number of two-choice SAW loops (C_N) with perimeter length N from $N=4$ to 52 at intervals of 4.

N	C_N
4	2
8	0
12	6
16	16
20	90
24	432
28	2 156
32	10 944
36	56 304
40	293 320
44	1 545 676
48	8 226 384
52	44 164 250

Separate extrapolations of μ_N for $x=1.5$ and 1.0 against $1/N$ as $1/N\to 0$ give us $\mu = 1.565\pm 0.002$. To calculate the exponent h we defined h_N as

$$h_N = \frac{\log(C_{N+4}/C_N) - \log(\mu_{N+4}^{N+4}/\mu_N^N)}{\log[(N+4)/N]}.$$

Again separate extrapolations of h_N for $x=1.5$ and $x=1.0$ against $1/N$ as $1/N\to 0$ give us $h=-1.5\pm 0.1$ (see figures 1(a) and 1(b)). This h value is consistent with the de Gennes relation.

To calculate the number of loops for the spiral SAW we first see that the spiral constraint itself restricts the loops to be of two categories, rectangular and non-rectangular types. A rectangular loop with some specified values of its side lengths will occur several times, depending on the position of the starting point on its perimeter. Taking into consideration this multiplicity, the number of rectangular spiral SAW loops will be $1+2+3+\cdots+n$, where for loops of perimeter length N, $n=\frac{1}{2}(N-2)$ resulting in a total of $(N-8)(N/8)$ such loops, when the first step direction is fixed.

An example of the non-rectantular spiral SAW loops for $N=14$ is shown in figure 2. The starting point is shown by a '×'. If the starting direction is fixed, the number of non-rectangular loops which corresponds to a rectangle of lengths a and b is $(a-1)(b-1)$ which is the number of lattice points within the rectangle of sides a and b. Therefore the number of non-rectangular loops with perimeter length N, with the starting direction fixed, is

$$n\times 1+(n-1)\times 2+(n-2)\times 3+\cdots+1\times n$$

where for N-step loops n is $(N-6)/2$ resulting in

$$\tfrac{1}{48}(N-2)(N-4)(N-6)$$

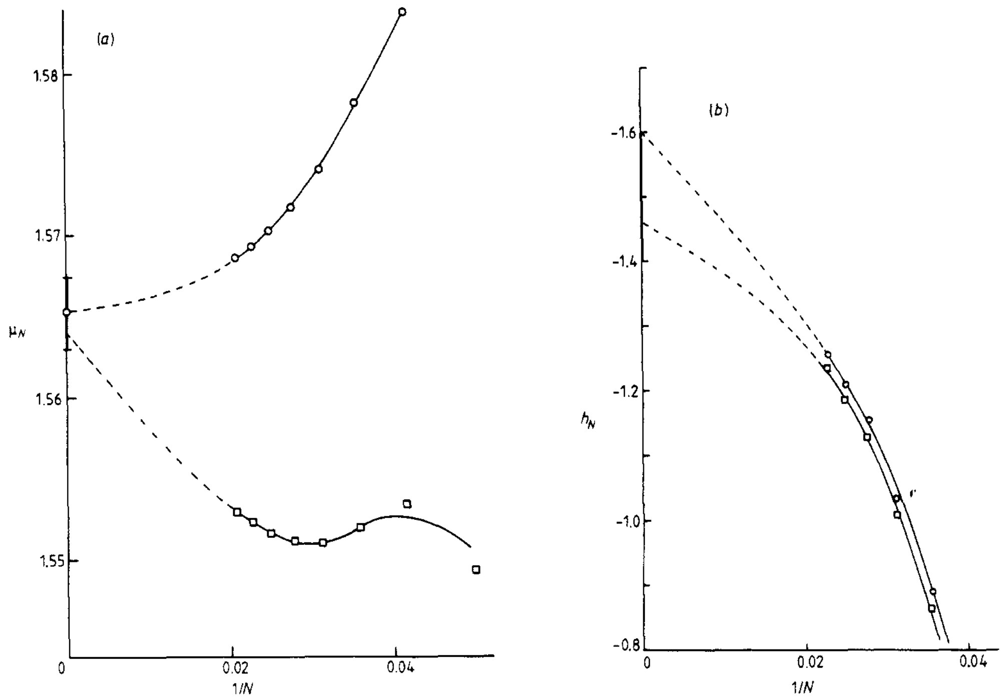

Figure 1. (a) Plot of μ_N against $1/N$ (○, $x=1.5$; □, $x=1$). (b) Plot of h_N against $1/N$ (○, $x=1.5$; □, $x=1$).

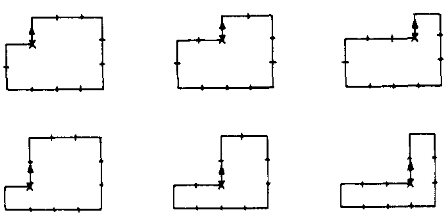

Figure 2. Different types of non-rectangular spiral SAW loops when the corresponding rectangle has dimensions 4 and 3.

for the total number of such walks. The total number of spiral SAW loops, which is the sum of rectangular and non-rectangular loops, is thus

$$\tfrac{1}{48}(N-2)(N^2-4N+24)$$

when the starting direction is fixed. This relation shows that for spiral SAW loops $\mu=1$ and $h=3$.

According to de Gennes' magnetic mapping, $h=-d\nu$ for ordinary SAW loops, where ν is the SAW average end-to-end distance exponent and d is the dimensionality of the lattice. The two-choice SAWs belong to the same universality class as ordinary SAWs and so $\nu=0.75$ for $d=2$ (Grassberger 1962, Manna and Chakrabarti 1984). The result $h=-1.5\pm0.1$ for two-choice SAW loops supports the fact that two-choice restriction on the SAW does not change its universality class. For spiral SAWs on the square lattice

$\nu = 0.50$ (Blöte and Hilhorst 1984). The result $h = 3$ for the spiral SAW loop is consistent with the fact that spiral SAWs do not belong to the universality class of ordinary SAWs. However, it is to be noted that for spiral SAWs the scaling relation $h = -d\nu$ is not obeyed.

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